Experimental scheme for unambiguous discrimination of linearly independent symmetric states

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We propose an experimental setup for discriminating four linearly independent nonorthogonal symmetric quantum states. The setup is based on linear optics only and can be configured to implement both optimal unambiguous state discrimination [Chefles and Barnett, Phys. Lett. A 250, 223 (1998)] and minimum error discrimination. In both cases, the setup is characterized by an optimal success probability. The experimental setup can be generalized to the case of discrimination among $N$ linearly nonorthogonal symmetric quantum states. We also study the discrimination between two incoherent superpositions of symmetric states. In this case, the setup also achieves an optimal success probability in the case of unambiguous discrimination as well as minimum error discrimination.

I. INTRODUCTION

Maximally entangled quantum states are at the core of several protocols in the field of quantum communications, such as quantum teleportation [1], entanglement swapping [2], and dense coding [3]. A common stage of these protocols is the projection of two quantum systems onto Bell states. Thereafter, a transformation is applied conditional on the projection’s result. The use of partially entangled states leads to a reduction of the fidelity or to a lower efficiency of these protocols. In this case, the transformation to be applied depends on measurement results which involve the discrimination of nonorthogonal quantum states. This task cannot be carried out deterministically.

Nevertheless, it is possible to resort to known strategies for discriminating among nonorthogonal states, such as minimum error discrimination and unambiguous state discrimination. The simplest set of states is composed of two nonorthogonal states. In this case, the optimal success probability of the minimum error discrimination strategy is given by the Helstrom limit [4]. According to this strategy, we can distinguish between the two states with minimum probability of error by choosing appropriately the detection operators. For instance, if the states are pure and with equal $a$ priori probabilities the optimal measurement is a von Neumann measurement. On the other hand, the optimum unambiguous state discrimination for two pure states with equal $a$ priori probabilities is given by the Ivanovic-Dieks-Peres (IDP) limit [5]. In this strategy we detect the states without error but may get inconclusive results. In this case, the strategy is implemented using a generalized measurement. In this work, we show experimental schemes that achieve these limits.

Unambiguous state discrimination has been applied in the context of entanglement concentration [6], quantum teleportation of qudits [7], entanglement swapping of qudits [8], and dense coding [9]. It has been shown that under certain conditions the nonorthogonal states to be distinguished form a set of linearly independent symmetric states. The conclusive discrimination of this class of states allows one to conclusively teleport an unknown quantum state with unitary fidelity and with a given success probability. A similar result also holds in the cases of entanglement swapping and dense coding. It is in this context that the experimental implementation of discrimination schemes is important, since these schemes can help to improve the reliability of quantum-communication protocols.

Minimal error discrimination at the Helstrom limit was demonstrated via weak optical laser pulses where the two states to be discriminated were nonorthogonal states of polarization [10]. Unambiguous state discrimination of two nonorthogonal polarization states near the Ivanovic-Dieks-Peres limit [5] was experimentally implemented via weak optical pulses propagated through an optical fiber with polarization-dependent loss [11]. Unambiguous state discrimination at the IDP limit was also experimentally achieved via a free-space interferometer which allowed registration of conclusive and inconclusive events [12]. Unambiguous state discrimination has been experimentally investigated in the context of three nonorthogonal linearly independent states [13]. The discrimination of three nonorthogonal linearly dependent states has also been experimentally demonstrated [14].

Here we propose a feasible experimental setup for discriminating among a set of four nonorthogonal linearly independent symmetric states. The setup is based on linear optics and considers the processes required to generate, propagate, and discriminate the states. It can be configured to implement unambiguous state discrimination as well as minimum error discrimination. We show that in both cases the setup achieves the optimal success probability. The setup can be directly generalized to the discrimination among $N$ nonorthogonal linearly independent symmetric states. Thereafter, we study the discrimination of two mixed quantum states, which are incoherent superpositions of two states out of four nonorthogonal symmetric states. We show that the proposed setup can be modified to implement the optimal discrimination of these density matrices via both unambiguous state discrimination and minimum error discrimination.

This paper has been organized as follows. In Sec. II we review the problem of quantum state discrimination and the
main results concerning symmetric states. In Sec. III we determine the conditional unitary transformation necessary for discrimination in the case of four nonorthogonal symmetric states. In Sec. IV we describe an experimental setup for generating, propagating, and discriminating among the four nonorthogonal states. This setup is based on down-converted photons generated in a spontaneous parametric down-conversion (SPDC) process. We propose experimental discrimination schemes both for unambiguous discrimination and for minimum error discrimination. In Sec. V we study the two strategies for quantum state discrimination in the case of mixed states, which are incoherent superpositions of the symmetric states. Finally, in Sec. VI we summarize our results and describe the application of them to several quantum-communication protocols.

II. BRIEF REVIEW OF QUANTUM STATE DISCRIMINATION

Quantum state discrimination has been a renewed subject of study in recent years. Usually, this problem is introduced in the context of quantum communications. A sender resorts to a set of quantum states to transmit information. In order to access this information, the receiver proceeds to identify the states. However, this task can be carried out deterministically only if the set is composed of mutually orthogonal states. Otherwise, the discrimination can be carried out only with a certain success probability. The optimization of this success probability leads to different discrimination strategies.

Unambiguous quantum state discrimination arises when the condition is imposed that the states must be identified with certainty. This is possible at the expense of introducing an extra event which does not convey information about the state being discriminated. This is best described in the language of positive operator valued measurements (POVMs). Let us consider a set of quantum states \( \{ |\Psi_k\rangle \} \), with \( k = 0, \ldots, N-1 \) generated with \textit{a priori} probabilities \( \eta_k \). The discrimination strategy is based on the existence of operators \( A_k \) with \( k = 0, \ldots, N-1 \) and the operator \( A_I \) satisfying the relations

\[
A_I^* A_I + \sum_{k=0}^{N-1} A_k^* A_k = 1, \tag{1}
\]

where \( I \) denotes the identity of the initial Hilbert space, and

\[
p_{k,j} = \text{tr}(|\Psi_k\rangle\langle\Psi_k| A_j^* A_j) = p_k \delta_{k,j} \quad \forall \quad k, j = 0, \ldots, N-1, \tag{2}
\]

where \( p_k \) is the probability of discriminating without ambiguity the state \( |\Psi_k\rangle \). The total probability of an inconclusive discrimination is given by

\[
P_I = \sum_{k=0}^{N-1} \eta_k \text{tr}(|\Psi_k\rangle\langle\Psi_k| A_k^* A_k). \tag{3}
\]

The operators \( A_k \) with \( k = 0, \ldots, N-1 \) are constructed in such a way that the total unambiguous discrimination probability \( \sum_{k=0}^{N-1} \eta_k p_k \) is maximal. It has been shown that these operators exist only if the states in the set \( \{ |\Psi_k\rangle \} \) are linearly independent. This strategy can also be formulated in terms of unitary transformations and von Neumann measurements by enlarging the dimension of the Hilbert space from \( N \) to at most \( 2N-1 \). This is accomplished by introducing a two-dimensional ancillary system [15]. After coupling the ancilla to the quantum system, usually via a conditional evolution, a measurement on the ancilla projects the quantum system onto a state which depends on the result of ancilla’s measurement. One of the possible results will allow the conclusive discrimination of the original quantum state, and the other one gives an inconclusive measurement.

In this paper, we propose an experimental setup based on linear optics for discriminating nonorthogonal quantum states lying in \( (N=2^M) \)-dimensional Hilbert space. For the sake of simplicity, we describe an experimental setup in the case of dimension 4 \( (M=2) \), which can be directly generalized to larger-dimensional cases. This can also be extended to an arbitrary dimension by the optical implementation of the inverse Fourier transform on such a dimension. The setup considers the generation process, propagation, and discrimination of quantum states. We restrict ourselves to the case of nonorthogonal linearly independent states \( \{ |\Psi_0\rangle \} \) which are symmetric, defined by

\[
|\Psi_I\rangle = Z|\Psi_0\rangle, \tag{4}
\]

where \( |\Psi_0\rangle = \sum_{k=0}^{N-1} c_k |k\rangle \) is a normalized state, i.e., \( \sum_{k=0}^{N-1} |c_k|^2 = 1 \). The action of the Z operator on this state is such that \( Z|k\rangle = \exp(2\pi i k / N)|k\rangle \) and \( Z^N = I \). Following Ref. [16], the action of the conditional unitary evolution of a two-dimensional ancilla with the quantum system is written as

\[
U|\Psi_I\rangle \otimes |0\rangle_a = \sqrt{p_l} |a_l\rangle |0\rangle_a + \sqrt{1-p_l} |\phi_l\rangle |1\rangle_a. \tag{5}
\]

where the \( |0\rangle_a \) state is a known initial state of the ancillary system, and the states \( \{ |a_l\rangle \} \) and \( \{ |\phi_l\rangle \} \) are orthogonal states and linearly dependent states, respectively, of the quantum system. In the case of measuring an ancilla in the \( |0\rangle_a \) state, the \( |\Psi_I\rangle \) state is projected onto a \( |a_l\rangle \) state, with success probability \( p_l \), which allows a conclusive discrimination with a von Neumann measurement in the basis \( \{ |a_l\rangle \} \), since these states are orthogonal. In the case of the outcome \( |1\rangle_a \) for the ancilla, the state of the system is projected onto linearly dependent states \( \{ |\phi_l\rangle \} \), which cannot be unambiguously discriminated. In this process, the optimal conclusive probability to discriminate among a set of \( N \) nonorthogonal symmetric states is \( P_{\text{opt}} = N|c_{\text{min}}|^2 \) [16], where \( |c_{\text{min}}| \) is the smallest coefficient, i.e., \( |c_{\text{min}}| \leq |c_l| \), of state \( |\Psi_I\rangle \) for \( l = 0,1,\ldots,N-1 \).

III. SYSTEM-ANCILLA CONDITIONAL EVOLUTION

Here, we consider the case of four nonorthogonal linearly independent symmetric states, which are denoted by \( \{ |\Psi_0\rangle, |\Psi_1\rangle, |\Psi_2\rangle, |\Psi_3\rangle \} \). These states are generated by applying the unitary transformation \( Z^l \) on the \( |\Psi_0\rangle \) state, such that \( |\Psi_l\rangle = Z^l |\Psi_0\rangle \), with \( l = 0,1,2,3 \). The \( |\Psi_0\rangle \) state is defined by
where \( c_k \) coefficients obey the normalization condition and are considered to be real. In general, these coefficients can be written as \( c_0 = \cos \theta_1, \ c_1 = \cos \theta_2 \sin \theta_1, \ c_2 = \cos \theta_3 \sin \theta_2 \sin \theta_1, \) and \( c_3 = \sin \theta_3 \sin \theta_2 \sin \theta_1. \) The convenience of this parametrization becomes clear later on, when we discuss the physical implementation of the discrimination protocol. For building up the conditional unitary evolution, we will make use of the general approach proposed by He and Bergou [17], which allows one to find a transformation that projects the \( |\Psi_i\rangle \) states onto a set of orthogonal states \( \{|\phi_i\rangle\} \) and onto another set of linearly dependent states \( \{|\phi_i\rangle\} \). First, we obtain the diagonal form of the \( A_i^\dagger A_i \) operators; this can be done when there exists a unitary operator \( U_0 \) acting on the initial Hilbert space, which gives

\[
U_0 A_i^\dagger A_i U_0 = \sum_{k=0}^{D-1} \lambda_k |\alpha_k\rangle\langle \alpha_k|,
\]

where \( |\alpha_k\rangle \) is an eigenvector of the \( A_i^\dagger A_i \) operator with eigenvalue \( \lambda_k. \) Since the \( A_i^\dagger A_i \) operator is positive, its eigenvalues are defined between zero and one, and therefore we can define Hermitian operators

\[
A_i^\dagger = A_i = U_0^\dagger \sum_{k=0}^{D-1} \sqrt{\lambda_k} |\alpha_k\rangle\langle \alpha_k| U_0,
\]

\[
A_i^\dagger = A_i = U_0^\dagger \sum_{k=0}^{D-1} \sqrt{1 - \lambda_k} |\alpha_k\rangle\langle \alpha_k| U_0.
\]

The unitary transformation, in the enlarged ancilla-system space, takes the following form:

\[
U = \begin{pmatrix} A_s & -A_t \\ A_t & A_s \end{pmatrix},
\]

where \( A_i^\dagger A_i = \sum_{k=0}^{D-1} \lambda_k^2 |\alpha_k\rangle\langle \alpha_k| \) is the operator corresponding to a conclusive result. The \( U \) operator is not unique; there are three other similar forms [17]. We have assumed a qubit ancilla, with basis \( \{|0\rangle, |1\rangle\} \) and initially prepared in the state \( |0\rangle. \) After the conditional evolution of the composite ancilla-system, the measurement on the ancilla gives the state \( |0\rangle \) determines the action of the \( A_i^\dagger A_i \) operator on the original quantum system, so that the discrimination process is conclusive. In the other case, the measurement on the ancilla is \( |1\rangle, \) the POVM element \( A_i^\dagger A_i \) had acted on the quantum system and hence the discrimination process fails. An explicit form for the \( A_k \) operator was found by Chefles [6],

\[
A_k = \frac{\sqrt{p_k}}{\langle \Psi_k^\perp | \Psi_k^\perp \rangle} |\psi_k\rangle\langle \psi_k|,
\]

\[
|\psi_k\rangle = \sqrt{p_k} |u_k\rangle,
\]

where the \( |u_k\rangle \) states form an orthonormal basis for \( \mathcal{H}; \) \( |\Psi_k^\perp\rangle \) are the reciprocal states; and \( p_k \) is the probability to get the \( k \)-th outcome. This operator is consistent with

\[
A_k |\psi_k\rangle = \sqrt{p_k} |u_k\rangle.
\]

The reciprocal states \( |\Psi_k^\perp\rangle \) are defined by

\[
|\Psi_k^\perp\rangle = \frac{1}{\sqrt{q}} \sum_{r=0}^{N-1} e^{i\pi m r k} |r\rangle,
\]

where \( q = N \) [16]. These states are also linearly independent and symmetric with respect to the \( Z \) transformation. Then operators \( A_i \) and \( A_j \) in the case of discriminating the \( \{|\Psi_i\rangle, |\Psi_j\rangle, |\Psi_z\rangle, |\Psi_3\rangle\} \) states are

\[
A_s = \sin \theta_3 \sin \theta_2 \tan \theta_1 |0\rangle\langle 0| + \sin \theta_3 \tan \theta_2 |1\rangle\langle 1| + \sin \theta_2 |2\rangle\langle 2|,
\]

and

\[
A_t = \sqrt{1 - \sin^2 \theta_3 \sin^2 \theta_2 \tan^2 \theta_1} |0\rangle\langle 0| + \sqrt{1 - \sin^2 \theta_3 \sin^2 \theta_2} |1\rangle\langle 1| + \sqrt{1 - \tan^2 \theta_2 |2\rangle\langle 2|}.
\]

Here, we have assumed all \( a \) \textit{priori} probabilities \( \eta_i \) to be equal, with a value \( 1/N, \) and the discrimination probabilities to be \( p_k = p_D [16]. \)

After applying the conditional evolution on the composite ancilla-system, we get

\[
U |\psi_i\rangle \otimes |0\rangle = \sqrt{p_D} |u_i\rangle |0\rangle + \sqrt{1 - p_D} |\phi_i\rangle |1\rangle,
\]

such that the symmetric states \( \{|\Psi_0\rangle, |\Psi_1\rangle, |\Psi_2\rangle, |\Psi_3\rangle\} \) are projected to \( \{|u_0\rangle, |u_1\rangle, |u_2\rangle, |u_3\rangle\} \) with a probability \( p_D = |e_{\text{min}}|^2 \) when a projective measurement on the ancilla gives the state \( |0\rangle, \) where \( e_{\text{min}} = \min\{|\cos \theta_1\rangle, |\cos \theta_2 \sin \theta_1\rangle, |\cos \theta_3 \sin \theta_2 \sin \theta_1\rangle, |\sin \theta_3 \sin \theta_2 \sin \theta_1\rangle\}. \) For instance, in the case of angles satisfying \( 0 \leq \theta_1 \leq \pi/3, \) \( 0 \leq \theta_2 \leq 0.3 \pi, \) and \( 0 \leq \theta_3 \leq \pi/4, \) the minimum coefficient is \( |\sin \theta_1 \sin \theta_2 \sin \theta_3| \).

In this case, the orthogonal states \( |u_i\rangle \) are found to be the four-dimensional Fourier transform acting on logical states \( |\rangle \), i.e., these states are given by

\[
|u_i\rangle = \mathcal{F} |\rangle = \frac{1}{2} \sum_{k=0}^{3} e^{i\pi k i/2} |k\rangle.
\]

Hence, the orthogonal states \( |u_i\rangle \) are superpositions of the logical basis. We must apply the inverse of the Fourier transform to carry out the discrimination among them in the logical basis which, in its matrix representation is given by

\[
\mathcal{F}^{-1} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 & 1 & 1 \\
1 & -i & -1 & i \\
1 & -1 & 1 & -1 \\
i & -1 & -1 & i \end{pmatrix}.
\]

In terms of linear optics, this transformation can be regarded as a symmetric eight-port beam splitter [18].

**IV. EXPERIMENTAL SETUP WITH TWO-PHOTON STATES**

It is possible to implement the discrimination protocol by using single-photon states, where the logical states are
fined by propagation paths. However, having a controlled source of single photons is rather difficult. Usually, for this purpose a highly attenuated pulsed laser is used, with a mean photon number less than one photon per pulse. For instance, ultralow intensity pulses are used for establishing quantum key distributions in cryptography experiments [19].

A. Unambiguous discrimination

Here, we describe an experimental setup for implementing the optimal protocol for discriminating linearly independent quantum states, by using a simple optical system based on two-photon states generated in a spontaneous parametric down-conversion process. The optimum is defined in the sense that the protocol maximizes the average success probability. We codify nonorthogonal quantum states in propagation paths in one of the down-converted photons (signal) and the other down-converted photon (idler) that will be used for coincidence measurement, i.e., this photon will ensure the presence of the other photon in one of the nonorthogonal states. Thus, logical state $|j\rangle$, with $j=0, 1, 2, 3$, corresponds to the $j$th propagation path of the photon. The discrimination protocol is divided into four steps: preparation of the symmetric states; conditional ancilla-system evolution; projective measurement on the ancilla; and finally, in the case of conclusive measurement, identification of a state belonging to a set of mutually orthogonal states.

As we have described above, generalized quantum measurements are implemented by embedding the quantum system into a large Hilbert space by adding an ancilla followed by an entangling operation. In this protocol, we use the polarization degree of freedom of the photon as our ancillary system. Hence, in the preparation stage of the symmetric states, we consider a photon initially prepared with horizontal polarization as input. Using half-wave retardation plates (HWPs), a polarized beam splitter (PBS), and phase shifters (PSs) we are able to generate the four symmetric states; see Fig. 1. The HWP, rotates the polarization of the photon through an angle $\pi/2 - \theta_i$. Hence, the vertical polarization of the photons is reflected at the PBS, and this component is used for defining the $|\langle -1 \rangle\rangle$ logical state. The transmitted polarization goes through the HWP$_{i+1}$. Actually, it is well known that, by using a HWP, a lossless PBS, and a PS with appropriate parameters, any $U(2)$ transformation can be implemented [20]. Considering that we have chosen the values of rotation angles at the HWPs such that the minimum coefficient is $|\sin \theta_i \sin \theta_j \sin \theta_l|$, we generate the $|\Psi_{0}\rangle$ state. We remark that HWP$_4$ rotates the polarization of path 4 from horizontal to vertical polarization, so that at the end of the preparation stage the polarization of the photon is factorized from the path states, i.e., in all the propagation paths the polarization remains vertical. In the same way, other states $|\Psi_{j}\rangle$ are generated by inserting phase shifters (see Fig. 1).

The first step in the discrimination protocol is to apply the conditional evolution (10) on the symmetric states, which corresponds to a conditional rotation of the polarization (ancilla) depending on the propagation paths of the photon (logical states). Hence, the transformation is defined by its action on the $|\Psi_{j}\rangle$ states and on the ancilla in the $|0\rangle$ state:

$$U(|\Phi_{j}\rangle |0\rangle) = [c_0 \sin \theta_1 \sin \theta_2 \tan \theta_1 |0\rangle + (i')c_1 \sin \theta_1 \tan \theta_2 |1\rangle + (-1)'c_2 \tan \theta_3 |2\rangle + (-1)'c_3 |3\rangle]\langle 0|$$

which is implemented with HWP$_5$, HWP$_6$, and HWP$_7$. The optimum discrimination process is attained when we choose rotation angles at these HWPs as $\theta_5 = \cos^{-1}(\tan \theta_1 \sin \theta_2 \sin \theta_3)$, $\theta_6 = \cos^{-1}(\tan \theta_2 \sin \theta_3)$, and $\theta_7 = \cos^{-1}(\tan \theta_3)$. The projective measurement is implemented right after applying the conditional evolution given by Eq. (19). Here, this is done by inserting polarized beam splitters PBS$_{0}$, PBS$_{1}$, and PBS$_{6}$ in propagation paths 0, 1, and 2, respectively, as is depicted in Fig. 2. An inconclusive measurement is obtained when a photon with horizontal polarization is transmitted through any one of these PBSs. If this projective measurement gives a conclusive measurement (transmission of vertical polarization) that one of the $|\Psi_{j}\rangle$ states has been transmitted then, for full discrimination, we need to determine which one of these $|\Psi_{j}\rangle$ states has been transmitted.

Here, all the $|\Psi_{j}\rangle$ states are orthogonal superpositions of the propagation paths, and each one of them is univocally associated with one of the nonorthogonal states. Hence, the last step in this protocol is the measurement of these orthogonal states and, for this purpose, it is convenient first to implement a unitary transformation satisfying $|l\rangle = F^{-1}|u_l\rangle$, since in this case the discrimination is done by a detection of a photon propagating in path $l$. This unitary transformation is carried out using an eight-port interferometer [18]. Figure 3 illustrates the scheme including this unitary transformation and all the previous stages for the unambiguous discrimination among four symmetric quantum states. In this figure,
detections at detectors $D_1$, $D_2$, $D_3$ and $D_4$ are in one-to-one correspondence with the states $|\Psi_0\rangle$, $|\Psi_2\rangle$, $|\Psi_1\rangle$, and $|\Psi_3\rangle$, respectively.

The protocol described above is easily generalized to the case of $N=2^M$ symmetric states to be discriminated. In Table I we list the number of optical components as a function of $N=2^M$. This protocol can also be implemented in the case of other dimensions. This requires decomposing the inverse Fourier transform at dimension $N$ in terms of beam splitters and retardation plates. For instance, in Ref. [21], this is done for the case of $N=3$, which can be mapped onto linear optical components. In particular, the quantum Fourier transform for $2^3$ has been reported both in Ref. [22] using linear optics and in Ref. [23] using fiber optics.

We consider using an argon ion laser in a continuous wave operation, which pumps a β-barium borate (BBO) nonlinear crystal with a power of 350 mW. The laser is made to operate in a single-frequency mode at 351.1 nm. In addition, an interference filter of width 10 nm, centered around 351.1 nm, is inserted into the propagation path of the pump field. Hence, two-photon states with center frequencies at 702.2 nm are generated. The nonlinear BBO crystal has been cut for SPDC type II, i.e., the propagation paths of down-converted photons are noncollinear. We select signal (idler) photons linearly polarized in the horizontal (vertical) plane by inserting a Wollaston prism, with a high extinction rate.

In the propagation path of the signal photon, we insert the setup for implementing the discrimination protocol. We assume that all the PBSs have an extinction rate of 1000:1. Controlled rotations of polarization states are accomplished by using HWPs. In our case, the relative angle is adjusted to generate the appropriate coefficients $c_i$ of symmetric states, Eq. (6). The purpose of the presence of the PS appears to be evident after the projective measurement, due to the implementation of $F^{-1}$ unitary transformations for mapping $|u_i\rangle$ states onto $|f\rangle$ states.

The eight-port interferometer must be completely balanced and stabilized, where we deal with four interferometers in a Mach-Zehnder configuration. This can be done by a phase adjustment mechanism on mirrors $M_1$ to $M_4$. The angles and positions of these mirrors must be adjusted to optimize the interference fringes in the four output ports. For this purpose, the mirrors and BS$_0$ to BS$_4$ must be mounted on precision translation stages, allowing the relative phase between the arms of Mach-Zehnder interferometers to be accurately varied. This stabilization process will be crucial for the discrimination protocol [24]. Here, we would like to remark that detectors $D_1$ to $D_4$ in the signal path are connected to detector $D_1$ in the idler path for coincidence measurement.

### B. Minimum error discrimination

The problem of discrimination among symmetric mixed states, with minimum error, is studied in Ref. [24]. The op-
timal strategy is obtained by determining the set of projection operators defined by
\[ \pi_k = \Phi^{-1/2}(\eta_k|\psi_k\rangle\langle\psi_k|)\Phi^{-1/2}, \]
where \( \Phi \) is the density operator formed by an incoherent superposition of the symmetric states \( |\psi_k\rangle \), with \textit{a priori} probabilities \( \eta_k \). As in the case of unambiguous discrimination, we also consider equal \textit{a priori} probabilities, i.e., \( \eta_k = 1/4 \). By replacing the symmetric states in Eq. (20), we get that \( \Phi = \sum_{k=0}^{3} \eta_k|\psi_k\rangle\langle\psi_k| \) and \( \Phi^{-1/2} = \sum_{k=0}^{3} (1/c_k)|\psi_k\rangle\langle\psi_k| \). Then, projection operators in the above expression are given by
\[ \pi_k = |u_k\rangle\langle u_k|, \]
with the \( |u_k\rangle \) states defined in Eq. (17), which are Fourier-transformed logical states. The probability of error in the minimum error discrimination protocol is
\[ P_{\text{ME}}^{\text{opt}} = 1 - \frac{1}{4}(c_0 + c_1 + c_2 + c_3)^2. \]

The experimental setup for implementing this protocol can also be seen in Fig. 3. However, in this case stage II is not included and the PBSs in stage III must be replaced by mirrors. There are no changes in the preparation stage nor in the detection stage. In this case, detections at detectors \( D_1, D_2, D_3, \) and \( D_4 \) are in one-to-one correspondence with the states \( |\Psi_0\rangle, |\Psi_3\rangle, |\Psi_3\rangle, \) and \( |\Psi_1\rangle \), respectively.

The probability of error and the probability of obtaining an inconclusive result satisfy the following relation:
\[ P_{\text{ME}} = 1 - P_D, \]
\[ 1 - \frac{1}{4}(c_0 + c_1 + c_2 + c_3)^2 \leq 1 - 4c_3^2, \]
when \( c_3 \) is the smaller coefficient.

V. DISCRIMINATION BETWEEN TWO MIXED QUANTUM STATES

In this section, we show how a modification of the discrimination setup can be used for discrimination between two mixed quantum states. As in the previous section, we also consider schemes for unambiguous discrimination and for minimum error discrimination. These mixed quantum states are incoherent superpositions of two symmetric states defined by
\[ \rho_+ = \frac{1}{2}(|\psi_0\rangle\langle\psi_0| + |\psi_2\rangle\langle\psi_2|), \]
\[ \rho_- = \frac{1}{2}(|\psi_1\rangle\langle\psi_1| + |\psi_3\rangle\langle\psi_3|). \]

The intersection between the supports of these mixed states is empty. Let us recall that the support is defined as the subspace spanned by the eigenvectors with nonzero eigenvalues [24,25] of an operator. These mixed states are called geometrically uniform states [26-28] due to the fact that they are connected by the \( Z \) unitary transformation, that is,
\[ \rho_- = Z\rho_+ Z^\dagger. \]

These mixed states have the following decomposition in the logical basis:
\[ \rho_\pm = \sum_{k=0}^{3} c_k^2|k\rangle\langle k| \pm c_0c_2(2)\langle 0| + 2\langle 0 |) \pm c_1c_3(|1\rangle\langle 3| + |3\rangle\langle 1|). \]

We assume that \( \rho_\pm \) mixed states are generated with equal probabilities, that is, \( \eta_\pm = \frac{1}{2}. \)

A. Unambiguous discrimination

In case of equal \textit{a priori} probabilities, the optimum probability of unambiguous discrimination of these mixed quantum states is given by [25,28]:
\[ P_{\text{opt}}^{\text{opt}} = 1 - F(\rho_+, \rho_-), \]
where \( F \) is the fidelity defined by \( F(\rho_+, \rho_-) = \text{tr}(\sqrt{\sqrt{\rho_+} \rho_- \sqrt{\rho_+}})^2 \). In this case of equal \textit{a priori} probabilities, the optimal failure probability \( Q_{\text{opt}} \) is equal to the fidelity. For \( \rho_+ \) and \( \rho_- \) defined by (27) the optimum probability explicitly reads as
\[ P_{\text{opt}}^{\text{opt}} = 2(c_2^2 + c_3^2). \]

In order to generate the mixed quantum states \( \rho_+ \) and \( \rho_- \), we consider the setup depicted in Fig. 4. In this figure we have inserted, in each one of the propagation paths, an unbalanced interferometer in a Mach-Zehnder configuration. In the case of generating \( \rho_+ \) (\( \rho_- \)) phase shifts corresponding to states \( |\psi_0\rangle \) (|\psi_1\rangle) and \( |\psi_2\rangle \) (|\psi_3\rangle) are, respectively, inserted in the upper and lower arms of interferometers. Hence, after the interferometers the state generated is \( \rho_+ \otimes |0\rangle\langle 0| \) or \( \rho_- \otimes |0\rangle\langle 0| \), where \( |0\rangle\langle 0| \) is the initial state of the ancilla, in this case the vertical polarization of the photon.

If we send these mixed states through the discrimination stage in Fig. 3, the probability of unambiguous discrimination is equal to \( S=4c_3^2 \). However, this probability is less than the optimum \( P_{\text{opt}}^{\text{opt}} = 2(c_2^2 + c_3^2) \), since we have assumed that the \( c_k \) coefficients are decreasingly ordered. The optimal probability can be achieved with the help of a swapping operation between propagation paths 1 and 2. This operation maps the initial mixed states \( \rho_\pm \) onto the mixed states
\[ \tilde{\rho}_\pm = U_{12}\rho_\pm U_{12}^\dagger = p_1|\phi_{1_\pm}\rangle\langle\phi_{1_\pm}| + p_2|\phi_{2_\pm}\rangle\langle\phi_{2_\pm}|, \]
where \( |\phi_{1_\pm}\rangle = c_{1_\pm}|0\rangle \pm c_{1_\pm}|1\rangle \) and \( |\phi_{2_\pm}\rangle = c_{2_\pm}|2\rangle \pm c_{2_\pm}|3\rangle \); \( c_0^2 = c_0/|p_1|, c_2^2 = c_2/|p_1|, c_1^2 = c_1/|p_2|, c_3^2 = c_3/|p_2|, p_1 = c_0^2 + c_2^2 \), and \( p_2 = c_1^2 + c_3^2 \). \( U_{12} \) is the swapping operation between propagation paths 1 and 2. Thus, the swapping operation casts the initial mixed states into states with a block-diagonal form. In order to discriminate the mixed states \( \tilde{\rho}_\pm \), therefore, we can resort to the unambiguous discrimination of the pure states.
EXPERIMENTAL SCHEME FOR UNAMBIGUOUS... PHYSICAL REVIEW A 76, 062107 (2007)

\[
\pi_\pm = \Phi^{-1/2}(p_\pm \rho_\pm)\Phi^{-1/2}
\]

with

\[
\Phi = (p_+|\phi_{1+}\rangle\langle\phi_{1+}| + p_-|\phi_{1-}\rangle\langle\phi_{1-}|) = c_0^2|0\rangle\langle0| + c_2^2|0\rangle\langle0|,
\]

where \(p_\pm\) are the probabilities of the \(|\phi_{1\pm}\rangle\) states; here \(p_\pm = 1/2\). From these expressions we get that the projection operators are \(\pi_\pm = |\pm\rangle\langle\pm|\), with \(|\pm\rangle = (|0\rangle \pm |1\rangle)/\sqrt{2}\). The experimental setup is similar to the one depicted in Fig. 4 but stage II is not included and the PBSs in stage III must be replaced by mirrors. The error probability for these states is obtained from

\[
P_{\text{ME}}^{\text{opt}} = 1 - \frac{1}{2}\left(\text{tr}(\pi_+\rho_+^{\dagger}) + \text{tr}(\pi_-\rho_-^{\dagger})\right) = \frac{1}{2} - c_0^2c_2^2.
\]

Similarly, we obtain that \(P_{\text{ME}}^2 = \frac{1}{2} - c_0c_2 - c_1c_3\). (36)

If there is a photodetection at \(D_1\) or \(D_3\) (\(D_2\) or \(D_4\)) the state is identified as \(\rho_+ (\rho_-)\). However, these results are ambiguous, and the probability of having a wrong result is given by \(P_{\text{ME}}\), which coincides with the minimum error. This can be seen by computing the Helstrom limit [4], which is determined by computing the following expression:

\[
P_{\text{e}} = \frac{1}{2}(1 - \text{tr}[\eta_+\rho_+ - \eta_-\rho_-]).
\]

VI. SUMMARY

We have proposed a scheme for the experimental discrimination of four symmetric states. The protocol has been designed for obtaining the optimal value of conclusive measurements, which is given by Chebyshev’s bound. Our scheme considers a reduced number of optical components and it can easily be generalized to the case of \(2^N\) symmetric states. In the case of other dimensions, the protocol also works with the optical implementation of the inverse Fourier transform. This, to the best of our knowledge, is the first proposal that can be generalized to larger-dimensional quantum systems. The experimental setup is based on two-photon states from SPDC, which allows us to reach the optimal value for conclusive discrimination. The main experimental requirement is the stabilization of interferometers in Mach-Zehnder configurations. We have also considered modified versions of the above-mentioned setup for a minimum error discrimination; in this case our proposal also achieves the optimal value. Finally, we have described optical setups for discrimination between two density operators, which are incoherent superpositions of symmetric states. This was done both for the unambiguous discrimination and for the minimum error discrimination.
We envisage the employment of the setup described above, for discriminating nonorthogonal symmetric states, and for key distribution in a quantum cryptographic protocol. Recent works have demonstrated that cryptographic protocols [29,30] are more robust against noise channels when using larger-dimensional quantum systems [31]. For this purpose, the sender randomly chooses to generate one of the nonorthogonal states. In this case the propagation paths, after the generation stage, are coupled to single-mode fiber optics, so that the polarization remains constant throughout the fiber. The receiver implements the discrimination protocol and the cases of conclusive measurement give a common element of the key to both the sender and the receiver. The presence of an eavesdropper, in between the authenticated users, can be detected in the authentication stage, where user and receiver publicly announce a reduced number of the elements of the cryptographic key. Alternatively, this presence can also be noticed in a modification of the probabilities of nonconclusive measurements, which does not require a disclosing of part of the cryptographic key. This work is under study, and we will present elsewhere a study on the security of such a protocol. In addition, we will also study applying this protocol to the problem of discriminating between subsets of nonorthogonal quantum states; for this problem we will follow the work of Sun et al. [32], where the case of a subset from three nonorthogonal states is studied.

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